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Polynomial approximation of analytic functions on a finite number of continua in the complex plane

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Abstract

The Dzjadyk-type theorem concerning the polynomial approximation of functions on a continuum in the complex plane $\mathbb{C}$ is generalized to the case of polynomial approximation of functions on a compact set in $\mathbb{C}$ which consists of a finite number of continua.

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1. Introduction

Let $E \subset \mathbb{C}$ be a compact set with the connected complement $\Omega := \overline{\mathbb{C}} \setminus E$, where $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the extended complex plane. Denote $A(E)$ the class of all functions that are continuous on E and analytic in the interior of E. The case of empty interior is also considered. Let $\mathbb{P}_n$, $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$, be the class of complex polynomials of degree at most n. For $f \in A(E)$ and $n \in \mathbb{N}_0$, define

$$E_n(f, E) := \inf_{p_n \in \mathbb{P}_n} \|f - p_n\|_E,$$

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where $\|\cdot\|_E$ denotes the supremum norm on E. By the Mergelyan theorem [6, p. 339]:

$$\lim_{n\to\infty} E_n(f, E) = 0 \quad (f \in A(E)).$$

The behavior of $E_n(f, E)$ is closely related to smoothness properties of f and the geometrical structure of E. The most delicate part of this theory, known as Dzjadyk-type theorems, concerns pointwise estimates of the behavior of $|f(z) - p_n(z)|$ on the boundary $L := \partial E$ of E. We refer the reader to [6,16,13,4] and the many references therein for a comprehensive survey of this subject. We would like to observe that the overwhelming majority of Dzjadyk-type direct theorems are proved for the case when E is a continuum, i.e., Ω is simply connected. The case when Ω is multiply connected is discussed only in a few papers (cf. [10,11,14,15,9,3]). Each time the extension of a result from the case of a continuum to the case of a compact set uses quite specific and non-trivial constructions.

In this paper we show how this extension can be accomplished by using well-known Bernstein–Walsh lemma on the growth of a polynomial outside the compact set and the Walsh theorem on polynomial approximation of a function analytic in a neighborhood of a compact set with connected complement.

As a sample of a Dzjadyk-type theorem we use a recent result about simultaneous approximation and interpolation of functions on continua in the complex plane [5, Theorem 1].

2. Main results

In the sequel we denote $c, c_1, \ldots$ positive constants (possibly different in different occurrences) that may depend on parameters inessential to the argument.

First, let E be a continuum (with the connected complement $\Omega := \overline{\mathbb{C}} \setminus E$). The most general continua, for which the direct Dzjadyk-type theorems can be proved, form the class H^* [1] which is defined as follows. We say that $E \in H$ if any two points $z, \zeta \in E$ can be joined by an arc $\gamma(z,\zeta) \subset E$ whose length $|\gamma(z,\zeta)|$ satisfies the condition

$$|\gamma(z,\zeta)| \leqslant c\,|z-\zeta|, \quad c = c(E) \geqslant 1. \tag{2.1}$$

Let us compactify the domain Ω by prime ends in the Carathéodory sense [12]. Let $\tilde{\Omega}$ be this compactification, and let $\tilde{L} := \tilde{\Omega} \setminus \Omega$. Suppose that $E \in H$, then all the prime ends $Z \in \tilde{L}$ are of the first kind, i.e., they have singleton impressions $|Z| = z \in L$. The circle $\{\xi : |\xi - z| = r\}$, $0 < r < \frac{1}{2}\operatorname{diam}(E)$, contains one arc or finitely many arcs, dividing Ω into two subdomains: an unbounded subdomain and a bounded subdomain such that Z can be defined by a chain of cross-cuts of the bounded subdomain. Let $\gamma_Z(r)$ denote the arc whose unbounded subdomain is the largest for given Z and r. Thus, the arc $\gamma_Z(r)$ separates the prime end Z from ∞.

If $0 < r < R < \frac{1}{2}\operatorname{diam}(E)$, then $\gamma_Z(r)$ and $\gamma_Z(R)$ are the sides of the quadrilateral $Q_Z(r,R) \subset \Omega$ whose other two sides are the parts of L. Let $m_Z(r,R)$ be the module of this quadrilateral, i.e., the module of the family of arcs that separate the sides $\gamma_Z(r)$ and $\gamma_Z(R)$ in $Q_Z(r,R)$ [8, p. 133].

We say that $E \in H^*$ if $E \in H$ and there exist $c = c(E) < \frac{1}{2}\,\mathrm{diam}(E)$ and $c_1 = c_1(E)$ such that

$$|m_Z(|z-\zeta|, c) - m_{\mathcal{Z}}(|z-\zeta|, c) \leqslant c_1 \tag{2.2}$$

for any prime ends $Z, \mathcal{Z} \in \tilde{L}$ with their impressions $z = |Z|,\ \zeta = |\mathcal{Z}|$ satisfying $|z-\zeta| < c$.

In particular, H^* includes domains with quasiconformal boundary (see [8]) and the classes B_k^* of domains introduced by Dzjadyk [6]. For a more detailed investigation of the geometric meaning of conditions (2.1) and (2.2), see [2].

We study functions defined by their kth modulus of continuity ($k \in \mathbb{N} := \{1, 2, \ldots\}$). There is a number of definitions of these moduli in the complex plane (see [16]). The definition by Dyn'kin [7] is the simplest to explain. Set

$$D(z, \delta) := \{\zeta : |\zeta - z| \leqslant \delta\} \quad (z \in \mathbb{C},\ \delta > 0).$$

The quantity

$$\omega_{f,k,E}(\delta) := \sup_{z \in E} E_{k-1}(f, E \cap D(z, \delta)),$$

where $f \in A(E),\ k \in \mathbb{N},\ \delta > 0$, is called the k-th *modulus of continuity* of f on E. It is known (cf. [16, Chapter 5]) that the behaviour of this modulus for $E \in H$ is essentially the same as in the classical case of the interval $E = [-1, 1]$. In particular,

$$\omega_{f,k,E}(t\delta) \leqslant c\, t^k\, \omega_{f,k,E}(\delta) \quad (t > 1, \delta > 0). \tag{2.3}$$

Denote $w = \Phi_E(z)$ the function which maps Ω conformally and univalently onto $\Delta := \{w : |w| > 1\}$ and is normalized by the conditions

$$\Phi_E(\infty) = \infty,\ \ \Phi'_E(\infty) > 0.$$

Let

$$L_{\delta,E} := \{\zeta \in \Omega : |\Phi_E(\zeta)| = 1 + \delta\} \quad (\delta > 0),$$

$$\rho_{\delta,E}(z) := \mathrm{dist}(z, L_{\delta,E}) = \sup_{\zeta \in L_{\delta,E}} |z - \zeta| \quad (z \in \mathbb{C},\ \delta > 0).$$

Theorem 1. *Let $E = \cup_{j=1}^m E_j$ consist of $m \in \mathbb{N}$ disjoint continua $E_j \in H^*$, $f \in A(E)$, $k \in \mathbb{N}$, and let $z_1, \ldots, z_N \in E$ be distinct points. Then for any $n \in \mathbb{N}$, $n > N + k$, there exists a polynomial $p_n \in \mathbb{P}_n$ such that*

$$|f(z) - p_n(z)| \leqslant c_1\, \omega_{f,k,E_j}(\rho_{1/n,E_j}(z)) \quad (z \in \partial E_j,\ j = 1, \ldots, m),$$

$$p_n(z_l) = f(z_l) \quad (l = 1, \ldots, N),$$

with c_1 independent of n.

For $j = 1$, Theorem 1 is proved in [5, Theorem 1]. For $j > 1$, the theorem extends the results from [10,11,9] to more general compact sets and new classes of functions. However, the main advantage of Theorem 1 is its relatively simple proof which follows immediately from [5, Theorem 1], (2.3), the inequality

$$\rho_{2\delta,E_j}(z) \leqslant c\rho_{\delta,E_j}(z) \quad (z \in \partial E_j,\ \delta > 0)$$

(see [5, (2.1)]) and the following statement which is, in itself, of interest.

Theorem 2. *Let* $E = \cup_{j=1}^m E_j$ *consist of* $m \in \mathbb{N}$, $m \geqslant 2$, *disjoint continua* E_j, $f \in A(E)$, $\|f\|_E \leqslant 1$, *and let* $z_1, \ldots, z_N \in E$ *be distinct points. Let for any* $n > n_0 \in \mathbb{N}$ *and* $j = 1, \ldots, m$ *there be a polynomial* $p_{n,j} \in \mathbb{P}_n$ *such that*

$$|f_j(z) - p_{n,j}(z)| \leqslant \varepsilon_j\left(\frac{1}{n}, z\right) \quad (z \in \partial E_j),$$

$$p_{n,j}(z_l) = f_j(z_l) \quad (z_l \in E_j),$$

where $f_j := f|_{E_j}$ *is the restriction of f to* E_j, *and the function* $\varepsilon_j(\delta, z)$, $0 < \delta \leqslant 1$, $z \in \partial E_j$, *satisfies, for any* $j = 1, \ldots, m$ *and* $z \in \partial E_j$, *the properties*:

(i) $\varepsilon_j(\delta, z)$ *is monotonically increasing in* δ;
(ii) $|\varepsilon_j(\delta, z)| \leqslant 1 \quad (\delta \leqslant \delta_0 \leqslant 1)$.

Then for any $n \in \mathbb{N}$, $n > c_1(n_0 + 1/\delta_0)$ *there exists a polynomial* $p_n \in \mathbb{P}_n$ *such that*

$$|f(z) - p_n(z)| \leqslant \varepsilon_j\left(\frac{c_2}{n}, z\right) + c_3\, e^{-c_4 n} \quad (z \in \partial E_j,\ j = 1, \ldots, m),$$

$$p_n(z_l) = f(z_l) \quad (l = 1, \ldots, N),$$

where c_k, $k = 1, 2, 3, 4$, *depend only on E and the choice of points* $z_1, \ldots, z_N$.

3. Proof of Theorem 2

Denote $g_\Omega(z, \infty)$, $z \in \Omega$, the Green function of Ω with pole at ∞ (see [17]). We extend it continuously to E by setting $g_\Omega(z, \infty) = 0$ for $z \in E$, and consider sets

$$E_r := \{z \in \Omega : \ g_\Omega(z, \infty) < r\} \quad (r > 0).$$

Denote r_E to be the maximal positive number such that E_r consists of exactly m components for $r \leqslant r_E$. We also introduce another geometric characteristic of E as follows:

$$R_E := \max_{1 \leqslant j \leqslant m} \|\log|\Phi_{E_j}(\cdot)|\|_{\partial E}.$$

By the maximum principle for harmonic functions

$$\log|\Phi_{E_j}(z)| - g_\Omega(z, \infty) \quad (j = 1, \ldots, m),$$

considered in Ω, we have

$$r_E \leqslant R_E.$$

Now, let $j = 1, \dots, m$ be fixed. Consider the function

$$h_j(z) := \begin{cases} 1, & z \in E_j, \\ 0, & z \in E \setminus E_j. \end{cases}$$

This function can be extended analytically to E_{r_E}. Hence, by the Walsh approximation theorem [17, pp. 75–76] there is $\mu_0^* = \mu_0^*(E) \in \mathbb{N}$, such that for any $\mu > \mu_0^*$ there is a polynomial $q_{\mu,j}^* \in \mathbb{P}_\mu$ satisfying the inequality

$$\|h_j - q_{\mu,j}^*\|_E < e^{-\mu r_E/2}.$$

Therefore, the polynomial

$$q_{\mu,j}(z) = q_{\mu,j}^*(z) + \sum_{l=1}^{N} \frac{\omega(z)}{\omega'(z_j)(z - z_l)} (h_j(z_l) - q_{\mu,j}^*(z_l)),$$

where

$$\omega(z) := \prod_{l=1}^{N} (z - z_l),$$

is of degree at most $\max(\mu, N-1)$. This polynomial for $\mu > \mu_0 = \mu_0(E, z_1, \dots, z_N) > \mu_0^* + N - 1$ satisfies the following conditions:

$$\|h_j - q_{\mu,j}\|_E < c\, e^{-\mu r_E/2} < e^{-\mu r_E/3},$$

$$q_{\mu,j}(z_l) = h_j(z_l) \quad (l = 1, \dots, N).$$

Let $\mu > \mu_0,\ v > \mu_0(n_0 + 1/\delta_0)$. Consider the polynomial $s_{v+\mu,j} := p_{v,j} q_{\mu,j}$ (of degree at most $v + \mu$) and the function $\tilde{f}_j := f\, h_j$.

Note that

$$s_{v+\mu,j}(z_l) = \tilde{f}_j(z_l) \quad (l = 1, \dots, N).$$

Moreover, for $z \in \partial E_j$, we obtain

$$|p_{v,j}(z)| \leqslant |p_{v,j}(z) - f(z)| + |f(z)| \leqslant 2.$$

Therefore, for $z \in \partial E_j$, we have

$$\begin{aligned} |\tilde{f}_j(z) - s_{v+\mu,j}(z)| &\leqslant |f_j(z) - p_{v,j}(z)| + |p_{v,j}(z)| |h_j(z) - q_{\mu,j}(z)| \\ &\leqslant \varepsilon_j\left(\frac{1}{v}, z\right) + 2e^{-\mu r_E/3}. \end{aligned} \tag{3.1}$$

Next, for $z \in \partial E_k,\ k \neq j$, by the Bernstein–Walsh lemma (see [17, p. 77]) we have

$$\begin{aligned} |\tilde{f}_j(z) - s_{v+\mu,j}(z)| &\leqslant \|p_{v,j}\|_E \|h_j - q_{\mu,j}\|_E \\ &\leqslant 2e^{vR_E - \mu r_E/3} \leqslant 2e^{-\mu r_E/4}, \end{aligned} \tag{3.2}$$

if

$$\mu := \left[12\frac{R_E}{r_E}\nu\right] + 1,$$

where $[a]$ denotes the integral part of a.

Let

$$s_{\nu+\mu}(z) := \sum_{j=1}^{m} s_{\nu+\mu,j}(z).$$

Then

$$s_{\nu+\mu}(z_l) = f(z_l) \quad (l = 1, \ldots, N).$$

For $z \in \partial E_j$, according to (3.1) and (3.2),

$$|f(z) - s_{\nu+\mu}(z)| \leqslant \varepsilon_j\left(\frac{1}{\nu}, z\right) + 2me^{-\mu r_E/4}.$$

Let $n > c_1(n_0 + 1/\delta_0)$, where

$$c_1 := \frac{3\mu_0(r_E + 6R_E)}{r_E}.$$

We set

$$\nu := \left[\frac{nr_E}{2(r_E + 6R_E)}\right].$$

Then $\nu + \mu \leqslant n$, i.e., $p_n := s_{\nu+\mu} \in \mathbb{P}_n$ and for $z \in \partial E_j$,

$$\begin{aligned} |f(z) - p_n(z)| &\leqslant \varepsilon_j\left(\frac{1}{\nu}, z\right) + 2me^{-\mu r_E/4} \\ &\leqslant \varepsilon_j\left(\frac{c_2}{n}, z\right) + c_3\, e^{-c_4 n}, \end{aligned}$$

with

$$c_2 = 3\left(1 + 6\frac{R_E}{r_E}\right),$$

$$c_3 = 2m\, e^{3R_E},$$

$$c_4 = \frac{2R_E\, r_E}{r_E + 6R_E}.$$

This completes the proof of Theorem 2.

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